STATISTICAL PROBLEMS IN SNOW MECHANICS

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ABSTRACT

Different types of statistical treatments are appropriate for evaluating measurements of different physical properties of snow. For density measurements, the mean and standard deviation of the measurements are meaningful parameters. In the case of strength measurements, the mean is not a very useful parameter, and some type of extreme-value statistics should be used. Weibull statistics appear to be appropriate for tensile strength data, while the thread bundle statistics of Daniels appear appropriate for the evaluation of shear strength data.

Introduction

Snow is a variable material; since most of its variations are random, statistical methods are indispensable in evaluating its behavior. Until recently, the statistical techniques applied to snow have been fairly simple. They have largely consisted of the calculation of the mean and sometimes of the standard deviation or confidence intervals of a series of measurements. Generally the measurements show such large scatter that attempts to correlate the various properties of snow have been frustrated.

Statistical techniques are available which can aid in a much better understanding of the behavior of snow and the relationships among its properties. It is necessary to use different kinds of statistical presentations, however, to extract the needed information from a series of experimental measurements. The appropriateness of a particular statistical technique depends on the properties being measured and the information desired from the measurements. As examples, we will examine statistical techniques which may be applied to density, brittle tensile strength, and shear strength data. Since the presence of free water in snow adds complications, this discussion will be limited to dry snow, although most of the concepts can be applied to wet snow as well.

None of the arguments presented are intended to be exhaustive. They are, rather, first attempts in this field; if they find application it will certainly be with extensive additions and modifications.

Snow Density

Dry snow is an intricate network of interconnecting ice crystals surrounded by air. The physical properties of ice and air are very different, and the volume ratio of ice to air can vary by over an order of magnitude. It seems obvious that the physical properties of snow must be related in very basic ways to the relative volumes of ice and air in a sample. The easiest measure of relative volumes is the density of the snow sample.

The variability of snow density is highest on a microscopic scale. If we were to proceed through a volume of snow taking 0.1 mm^3 samples at random, we would find some samples that were pure ice and some that were pure air. The data would range over values different by a factor of 10^3, but their mean value would be a measure of the bulk density of the total volume under consideration. Later we will see that this simple relationship between large and small samples does not hold for all of the properties of snow.

If consideration is restricted to density variations within major layers, the minor layering can be treated as statistical variations within the major layers. A careful study of the coefficients of variation within various sample volumes would answer the question of whether or not the densities of the major layers in an avalanche starting zone can be adequately determined from samples taken in a smaller volume. No study has been carried out with this approach in mind, but some data are available from other studies.

Martinelli (1971) measured the densities of 84 pairs of samples, each pair taken within a major layer. Each pair of 0.5×10^{-3} m^3 samples was taken in close proximity so that the

variability between the members of each pair is a measure of the variability within sample volumes of about $2 \times 10^{-3} m^3$. The mean coefficient of variation between the samples was 3 percent. I have sampled approximately 1 m^3 volumes in 12 different snow layers using both $0.5 \times 10^{-3} m^3$ and $2.3 \times 10^{-3} m^3$ sample tubes. There was no difference between the results obtained with the different tubes, and the mean coefficient of variation was 10.8 percent. Leaf[1] found that, in volumes of 20 m^3, the coefficient of variation of water equivalent was 18 percent. Since the measurement of water equivalent includes both depth and density variations, we can conclude that the variability of snow density within a volume of tens of cubic meters is not much larger than the variability within a volume of 1 m^3.

Although the data are insufficient, it appears that samples taken from snowpits near avalanche starting zones will provide adequate measures of the densities of snow layers in the starting zones. It also appears that samples should be taken as widely spaced as possible within the pit to insure representative data.

Brittle Tensile Strength

There is an obvious relationship between the density of snow and its strength. Since the pores in snow cannot support any load, the entire load must be supported by the ice network. On a microscopic scale, tensile strength samples would range between zero, for air, and the maximum tensile strength of monocrystalline ice.

Unlike density, the tensile strength of a large volume of snow will not be the mean of a large number of random, microscopic subsamples. Problems arise in determining the distribution of stresses within a snow sample, but the major complication is that once part of the ice network fails, an increased load is thrown onto the remainder and the whole sample is very likely to fail. Thus, the strength of a large sample would not be the mean of the strengths distributed throughout the sample volume, but would be the minimum of the strengths distributed throughout the ice network. Put another way, we are of necessity sampling for an estimate of the minimum strength.

Suppose that the strength of a large volume ($\sim 1 m^3$) of snow could be measured, then the volume broken up into smaller volumes ($\sim 10^{-3} m^3$) (without disturbing the distribution of strengths) and the strengths of the smaller volumes measured. The strength of the larger volume would not be the mean of the strengths of the smaller volumes, but would be the extreme low value found among them.

Cracks progagate through the entire slab thickness, which normally has a mean of about 1 m (Perla 1971). Stress gradients parallel to the slope are scaled to the slab thickness (Perla 1971, Smith 1972) so that a volume of about 1 m^3 seems to be the appropriate volume for strength considerations. Obviously, volumes of this size would be extremely difficult to test. We are then faced with the problem of determining the strengths of these large volumes of snow from smaller samples.

Fortunately, this problem is not unique to snow. A considerable amount of work has been done on strength problems of the type where there is a significant variation in the strengths distributed throughout a volume. The arguments given above are essentially those of Weibull (1939) who considered the general problem of the strength of nonuniform materials. Strength theories of this kind have the graphic name "weakest link" theories, apparently from the statement of Pierce (1926): "It is a truism, of which the mathematical implications are of no little interest, that the strength of a chain is that of its weakest link."

Weibull (1939) proposed the cumulative distribution function

$$F(\sigma) = 1 - e^{-V \left(\frac{\sigma - \sigma_u}{\sigma_o} \right)^m}, \quad \sigma \geq \sigma_u \qquad [1]$$

where σ is the applied stress, V the volume, and σ_u, σ_o, and m are material constants. This distribution is truncated in that the probability of failure at stresses below σ_u is zero.

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[1]Personal communication with Charles F. Leaf, Rocky Mountain Forest and Range Experiment Station, Fort Collins.

Snow is seriously weakened by large voids within snow layers, which appear to be formed during snow deposition. I have observed voids of approximately 10^3mm^3. Some measure of the size and frequency of these larger voids can be gained from a comparison of centrifugal tensile tests performed using two different sample diameters. Keeler and Weeks (1967), Keeler (1969), and Martinelli (1971) performed a total of 592 tests using a sample 60 mm in diameter. They reported that about 10 percent of their samples showed zero strength. Some of the zero strengths may have been due to sample damage, but some undoubtedly resulted when a void cut entirely across the specimen. In contrast I have performed 338 tests using 120 mm samples without observing any zero strengths (Sommerfeld and Wolfe 1972). Thus, voids above 100 to 150 mm in diameter are very rare or nonexistent within snow layers, supporting the idea that the strength distribution is truncated.

Although the voids probably act as stress concentrators in their vicinity, the stress is supported by the surrounding material. When a sample tube cuts a void, part or all of the surrounding material is left behind and a spuriously low strength is measured. Thus, we have the problem that the most important values are seriously disturbed by the sampling technique. Furthermore, the number of very low strength elements in a large volume is so low that we have no assurance we have sampled the lowest strength in a given volume unless we sample the entire volume.

An alternative method of predicting the strength of a large volume of snow comes from Weibull's distribution (equation 1). If all but the few lowest sample values of snow strength (which may be erroneous as discussed above) in a limited density range are fitted to Weibull's distribution, the constants and particularly σ_u can be determined. At very large V the function $F(\sigma)$ (equation 1) jumps from zero at $\sigma = \sigma_u$ to almost 1 at stresses just above σ_u. Thus, σ_u should be the large-volume strength. Figure 1 shows the theoretical infinite volume curve in comparison with an experimental curve obtained with sample volumes of 2.3×10^{-3}m^3.[2] It appears that a volume of 1 m^3 would fall very close to the infinite volume curve.

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[2]The reduced stress (σ/σ_m) used in this plot eliminates the effect of density differences among the samples (Sommerfeld 1971).
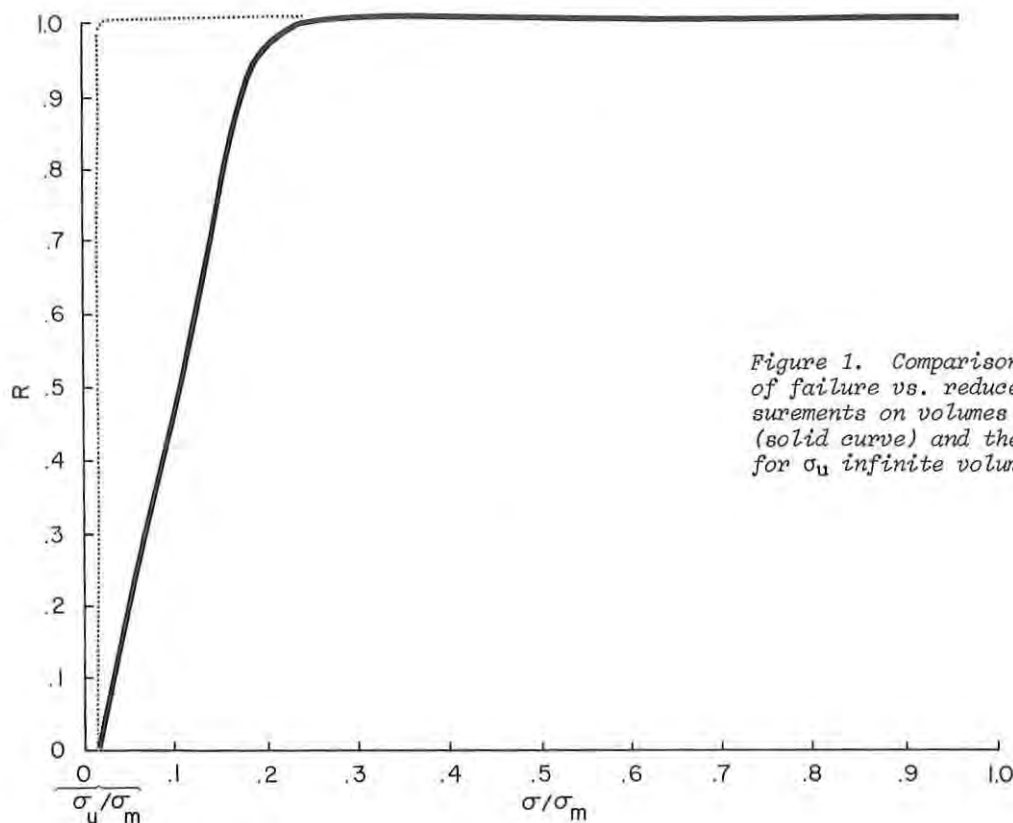

Figure 1. Comparison of the probability of failure vs. reduced stress for measurements on volumes of 2.3×10^{-3}m^3 (solid curve) and the theoretical curve for σ_u infinite volume.

To determine σ_u from centrifugal tensile tests, Weibull (1939) suggests the function

$$\ln \ln \frac{1}{1 - P} + \tfrac{1}{2} \ln \sigma = (m + \tfrac{1}{2}) \ln(\sigma - \sigma_u) + A(V) \qquad [2]$$

where P is the experimental probability of failure at stress σ and A(V) is a function of the sample volume. Equation 2 plots as a family of parallel curves, with different intercepts for different volumes. σ_u is determined as the unique value which gives the best fit to a straight line with $m + \tfrac{1}{2}$ as the slope.

Table I gives the calculated minimum strengths for various densities and types of snow. The snow was classified according to the scheme of Sommerfeld and LaChapelle (1970). Figure 2 shows σ_u plotted against mean density with the crystal type indicated at each point. In each case the lowest three strength values were discarded, since these values may be spuriously low, as previously discussed. In every case but the graupel (data group 11) the fit to the straight line was improved. In every case the fit to a straight line was very good, the lowest R^2 being 0.965 (fig. 3). A good fit to a straight line shows that the data fit the Weibull distribution very well. Of interest is the fact that the series which represents the normal course of equitemperature metamorphism (1, 2, 4, 5, 6) falls on a very good straight line in figure 2. Because case 3 was somewhat windblown and 9 was heavily windblown, they might be expected to show higher strengths for their densities due to mechanical reworking. Cases 7 and 8 are dry, temperature-gradient (TG) snow and 10 is wet TG snow. Here we see that the strength of a TG layer can vary over a large range, which is supported by field observations. Graupel (11) gives a lower strength, which also agrees with the observations of Perla (1971).

TABLE I

DENSITIES AND CALCULATED MINIMUM STRENGTHS (σ_u) FOR VARIOUS TYPES OF SNOW

Case Number	Snow Type	Ave. ρ Kgm^{-3}	σ_u dynes cm^{-2}
1	IA (P1e, P1f)	112	5.38×10^3
2	IIB2	254	14.24×10^3
3	IB (I 3aN2A)→IA1	128	9.53×10^3
4	IIA 1 − 2	147	9.32×10^3
5	IIA 1 − 2	228	14.15×10^3
6	IA (P2s)	73	4.12×10^3
7	IIIB3	281	15.58×10^3
8	IIIB 2 − 3	242	4.04×10^3
9	IB (I1)→IIB2	289	36.81×10^3
10	IIIB3→IVA1	331	20.69×10^3
11	IA (IR3b, 4a, 4b)	249	5.60×10^3
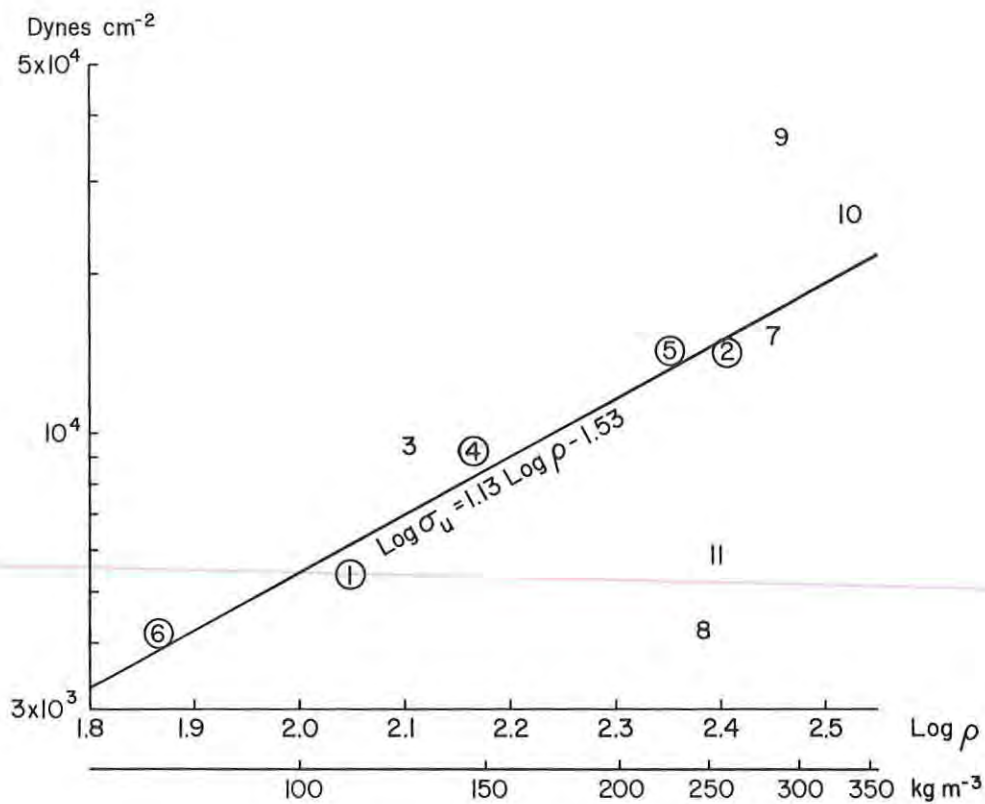
Figure 2. *Calculated minimum strengths (σ_u) vs. density for various types of snow.*

$$\text{Log}\, \sigma_u = 1.13\, \text{Log}\, \rho - 1.53$$

$R^2 = .965$

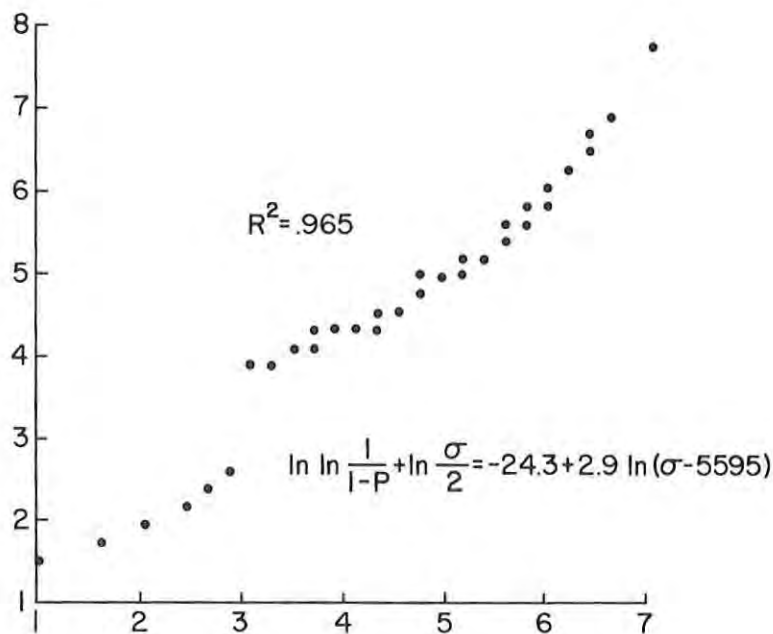$$\ln \ln \frac{1}{1-P} + \ln \frac{\sigma}{2} = -24.3 + 2.9 \ln (\sigma - 5595)$$

Figure 3. *The worst fit of the data (case 11) to equation 2.*

There are also similar but not so pronounced regularities exhibited by the other two con-
stants. However, whatever meaning might be ascribed to such regularities is still a matter of
conjecture.

Weibull's method as applied here boils down to assuming a distribution function (apparently
chosen by Weibull for mathematical tractibility) that determines a set of constants from a set of
data points, and uses the final function to extrapolate to an extreme limit. Undoubtedly such
extrapolations are risky, but until data are available on the large-volume strengths of snow, this
seems to be our only recourse. At least the Weibull analysis of the data leads to a fairly simple
representation of the brittle tensile strength of snow, and that representation is consistent with
other observations.

Shear Strength

Shear-strength measurements show characteristics similar to tensile-strength measurements
(fig. 4). There is very large scatter in the data, but the measurements appear to fall between
two extreme limits. Also, the larger the sample volume, the lower the mean of the measurements.[3]
Here again the statistics of Weibull could be applied to determine σ_u for the shear strengths of
various types of snow.

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[3]Personal communication with Ronald I. Perla, Rocky Mountain Forest and Range Experiment
Station, Fort Collins.

Figure 4. Shear strength vs. density.
(Original data supplied by R. I. Perla.)
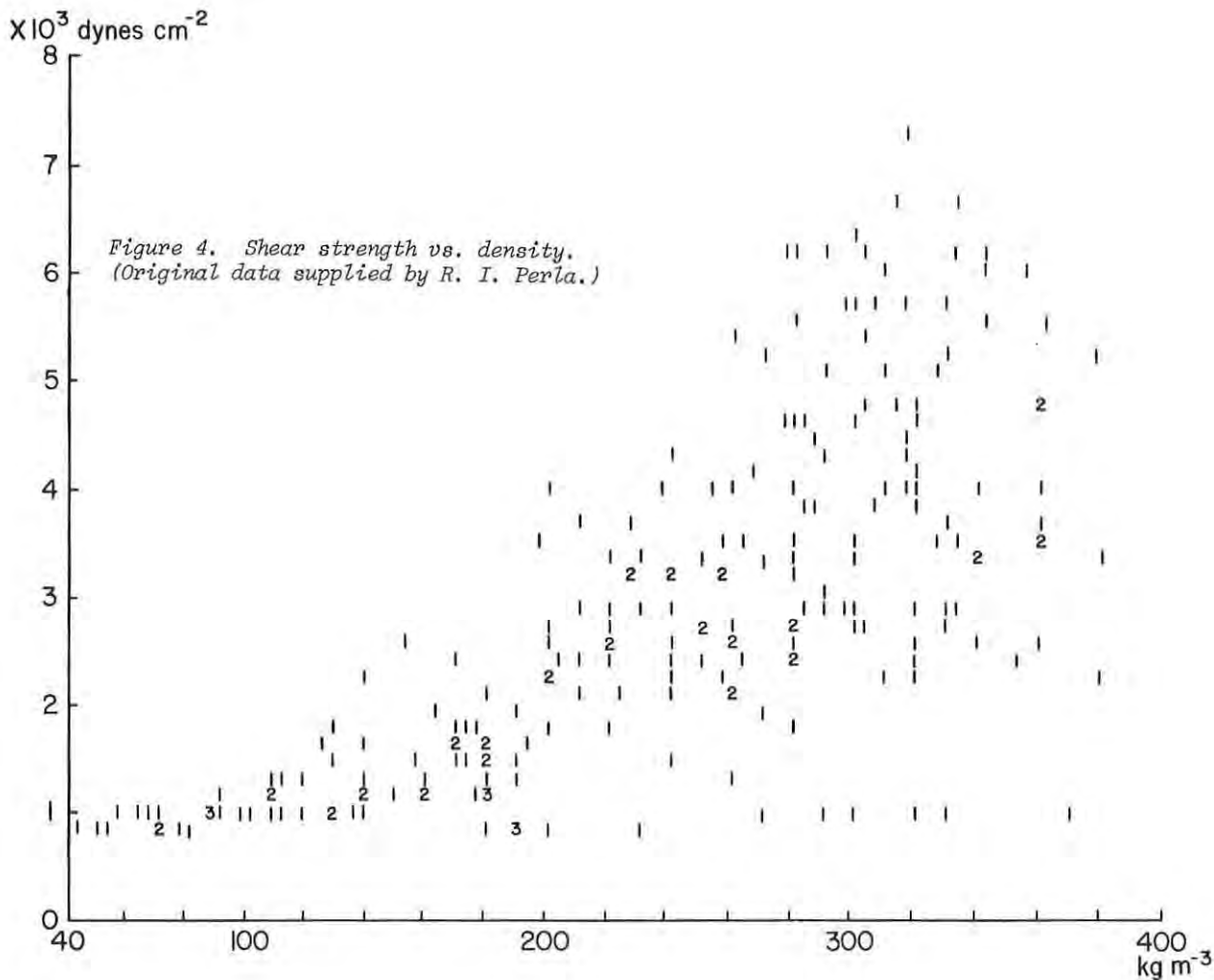
The assumption that avalanches are initiated by brittle shear crack propagation in the bed surface is open to question. During the formation of a tensile crack, the crack walls physically separate and the crack can become unstable and propagate in the manner of a Griffith (1921) failure. During shear failure, the failing parts are in contact, and friction and crack healing due to new bond formation impede crack growth. For these reasons a quasi-failure (Perla and LaChapelle 1970) is the likely initiating event. There is a strong possibility that a failure in a small part of the bed surface would not propagate elastically, and that the whole bed surface would not fail. Under such circumstances Weibull's statistics are not applicable, and the work of Daniels (1945), who considered the strengths of bundles of threads, appears to apply. Clearly the failure, at a particular stress, of one thread of a bundle may or may not lead to failure of the entire bundle depending on the strengths of the rest of the threads. Daniels showed that if $\theta(\sigma)$ is the probability density of the breaking stress (σ) of n elements, then a total load (S) is related to the load on each surviving thread (s) by

$$\frac{S}{n} = s \int_s^\infty \theta(\sigma) d\sigma \qquad [3]$$

and total failure occurs at the maximum

$$\frac{d\left[\frac{S}{n}\right]}{ds} = 0 = \frac{d}{ds} \left\{ s\int_s^\infty \theta(\sigma) d\sigma \right. \qquad [4]$$

s_T, the value of s at failure, can be found from the probability density function, and S_T/n, the stress at failure, can then be calculated.

The probability density function for snows of different densities can be derived from Perla's measurements.

The shear strengths measured by Perla for densities between 250 and 300 kg m^{-3} were found to fit the normal distribution with a mean of 30.0×10^3 dynes cm^{-2} and a standard deviation of 15.0×10^3 dynes cm^{-2}. Performing the calculations as in equations 3 and 4, we see that the failure stress for an entire layer within this density range, under our assumptions, is 15.8×10^3 dynes cm^{-2}, about 53 percent of the mean value.

Conclusions

The release of a slab avalanche is a complicated process. It appears that the initiation involves a volume of snow in the range 1 to 10 m^3. Field work indicates that it is necessary to interact with a volume of this size, with explosives, skiis, snowshoes, etc., to initiate an avalanche. Most slabs are about 1 m thick, and stress gradients parallel to the slope are scaled to the slab thickness, again indicating that a few cubic meters is the volume of interest for predicting slab failure.

Density is an important characteristic of snow slabs. The stresses within the slab are a direct result of the snow weight, and the mechanical properties of snow are related in basic ways to the snow density. Density is determined by the mean value of representative samples taken within a major layer. Available measurements are not conclusive, but do indicate that the density of 1 m^3 of snow in a major layer adequately represents the density of that layer.

The failure at the crown of a slab avalanche is undoubtedly brittle tensile fracture (Sommerfeld 1969), but the brittle tensile strength of snow cannot be understood in such simple terms as the density. The relationship of tensile strength to density is clarified by the application of "weakest link" strength theories. In particular, the statistics of Weibull (1939) give a consistent picture of the strength of a large volume of snow. It appears that the volume of interest in avalanche prediction (1 to 10 m^3) is large enough to be considered an infinite volume, and in that case the parameter σ_u in the Weibull distribution (equation 1) should be the snow strength. This strength has been shown to be a function of density for snows which have followed the normal course of equi-temperature metamorphism. Heavily windblown snow and temperature gradient snow need further study before their strength relationships become clear.

An important point which has been ignored in this study is the significance of the very large voids caused by rocks and vegetation which penetrate the slab. I know of no work on this subject, so at the moment nothing can be said concerning its importance.

The statistics of the shear strength of snow appear even more complicated. Since the failure of a part of the bed surface may or may not initiate the failure of the entire surface, the statistics developed for the evaluation of the strengths of thread bundles may be appropriate. In a limited test, measured shear strengths (in the density range 250 - 300 kg m^{-3}) were found to fit a normal distribution. Application of the statistics of Daniels (1945) for the large-volume case indicated that the bed surface should fail at about 53 percent of the mean shear strength of snow in that density range.

A complication which was ignored in the analysis is the time dependence of the shear strength of a snow layer. If a small part fails without initiating total failure, the snow in that part can rebond and the failure heal. This rate process might be included by making the shear strength distribution time dependent, but this has not been attempted.

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